AGT and the Segal-Sugawara construction

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Abstract

The conjectures of Alday, Gaiotto and Tachikawa [3] and its generalizations have been mathematically formulated as the existence of an action of a W-algebra on the cohomology or K-theory of the instanton moduli space, together with a Whitakker vector [7, 19, 29]. However, the original conjectures also predict intertwining properties with the natural higher rank version of the "Ext^1 operator" which was previously studied by Okounkov and the author in [10], a result which is now sometimes referred to as AGT in rank one [2, 27]. Physically, this corresponds to incorporating matter in the Nekrasov partition functions, an obviously important feature in the physical theory. It is therefore of interest to study how the Ext^1 operator relates to the aforementioned structures on cohomology in higher rank, and if possible to find a formulation from which the AGT conjectures follow as a corollary. In this paper, we carry out something analogous using a modified Segal-Sugawara construction for the $\hat{\mathfrak{sl}}_2\mathbb{C}$ structure that appears in Okounkov and Nekrasov's proof of Nekrasov's conjecture [25] for rank two. This immediately implies the AGT identities when the central charge is one, a case which is of particular interest for string theorists, and because of the natural appearance of the Seiberg-Witten curve in this setup, see for instance Dijkgraaf and Vafa [11], as well as [16].

1   Introduction

In [3], Alday, Gaiotto and Tachikawa proposed a collection of identities of explicit power series under a change of variables, each one associated a Riemann surface Σ of genus $g = 0, 1$, and a list of N marked points. On one side

of the equation are certain functions due to Nekrasov from four-dimensional supersymmetric gauge theory, which are defined as generating functions of equivariant localization integrals over a smooth noncompact complex algebraic variety $\mathcal{M}_{r,n}$ called the instanton moduli space, with respect to a certain torus action defined below. Here r, n are nonnegative integers called the rank and instanton number, or the rank and second Chern class in the description of $\mathcal{M}_{r,n}$ as a moduli space of framed sheaves on $\mathbb{CP}^2$. See [23] for a thorough introduction to this variety from several different descriptions. On the other side are correlation functions from Liouville theory, defined in terms of certain intertwiners of lowest weight representations of the Virasoro algebra [30]. One of the variables in these identities is the central charge c of these representations on the Liouville side, which is related to the torus parameters on the Nekrasov side, as in (20).

In an analogous way to the Hilbert-Chow map, the space $\mathcal{M}_{r,n}$ is a resolution of a singular space $\mathcal{M}_{r,n}^0$ which contains the moduli space of certain energy minimizing solutions to the four-dimensional Yang-Mills equation (instantons) modulo gauge symmetry, see [4, 12, 23]. These solutions are important in gauge theory because lowest energy solutions are expected to dominate the contributions to certain path integrals. In supersymmetric gauge theory, this approximation is sometimes exact, through a principle called supersymmetric localization, see [14] for a general introduction. Nekrasov's functions provide a mathematically rigorous definition of such integrals [24]. A precise mathematical conjecture supporting this statement due to Nekrasov relates certain limits of these quantities to the Seiberg-Witten free energy, and was proved by Okounkov and Nekrasov in [25]. See Okounkov's ICM notes [26] for an exposition.

To physicists, the AGT relations reflect the existence of a six dimensional superconformal quantum field theory on

$$X = \mathbb{R}^4 \times \Sigma,$$

predicted by the classification of superconformal field theories. It is expected that one can then recover the gauge theory partition function by letting Σ shrink away through a process called compactification, or dimensional reduction, see [31] as well as [28] for an introduction to the AGT conjectures. Mathematically, one would like to extend the AGT relations to more conceptual statements, which ideally would have implications to the physics. One direction that has been carried out by several authors [7, 19, 29] in much

greater generality for the case of pure (massless) gauge theory, is to identify the cohomology of $\mathcal{M}_{r,n}$ with a Hilbert space, and construct the relevant conformal symmetry representations, in a similar manner to Nakajima's earlier work [22]. In the case of AGT in rank two, these structures include an action of the Virasoro algebra on cohomology.

A more general version of the conjectures would have to incorporate the presence of matter, as is done in the original paper of AGT. Mathematically, this may be encoded in the Ext^1 operator W studied earlier by Okounkov and the author in the case of the Hilbert scheme of points on a general surface [10]. The main result of that paper identifies W with a "vertex operator" in terms of Nakajima's famous Heisenberg operators, which completely calculates the Nekrasov functions in rank one. This result was used in [2], to search for a basis of a Hilbert space on the Liouville side, which realizes the Ext^1 operator in the fixed point basis in rank two. An obviously desirable and more direct approach would be to formulate the cohomological structures of the last paragraph in a way that produces the intertwining properties with the Ext^1 operator predicted by AGT. Ideally, the original conjectures would then follow as a corollary.

In theorem 1, we do something similar in the setup of Nekrasov and Okounkov's proof of Nekrasov's conjecture [25]. In that paper, the authors used a representation of the affine special linear group $\hat{\mathfrak{sl}}_r\mathbb{C}$ to study the "dual partition function," which is a sort of generating function of the original partition functions. The main idea of this paper is to apply the Segal-Sugawara construction to the $\hat{\mathfrak{sl}}_r\mathbb{C}$ action for $r = 2$, and then to locate an extension of it that intertwines the transformed Ext^1 operator in the desired way. The correct choice of action turns out not to be the most obvious candidate, as seen in proposition 1 below. This is a fortunate occurence because the usual Segal-Sugawara construction would lead to non-irreducible representations of the Virasoro algebra, and would derail the argument.

In corollary 1, we show that this implies the AGT relations in the special case where the central charge is set to one, and Σ is a genus one Riemann surface with an arbitrary number of points removed. Specializing the central charge is simpler than the general "refined" case, but is of particular interest to string theorists. See for instance [11], which predicts a proof of AGT using matrix models for this case, as well as [16]. Along the lines of [11], the case of $c = 1$ and $\Sigma = \mathbb{CP}^1$ with four points removed was studied by Morozov, Morozov and Shakirov [20] using the Dotsenko-Fateev matrix model and Selberg integrals as the starting point on the Liouville side. The context

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of the dual-partition function is also particularly interesting because of the natural appearance of the Seiberg-Witten curve.

1.1 Acknowledgements

2 The AGT relations

2.1 Nekrasov functions

We now recall some mathematical background to define the Nekrasov functions. The definition of the moduli space of framed torsion free sheaves on $\mathcal{M}_{r,n}$ can be found in Nakajima's book [23] as well as [15]. For an introduction to Nekrasov's functions and their mathematical and physical meaning, see [24, 26].

Let $\{(x_0 : x_1 : x_2)\}$ coordinatize the complex projective plane $\mathbb{P}^2 = \mathbb{CP}^2$, and let $\mathbb{P}^1_\infty \subset \mathbb{P}^2$ be the line at infinity defined by $x_0 = 0$. The moduli space of framed torsion free sheaves is given set-theoretically by

$$\mathcal{M}_{r,n} = \{(F, \Phi) : \mathrm{rk}(F) = r, c_2(F) = n\} \tag{1}$$

where F is a torsion free sheaf on $\mathbb{P}^2$ which is locally free in a neighborhood of $\mathbb{P}^1_\infty$, and Φ is a choice of isomorphism

$$\Phi : F\big|_{\mathbb{P}^1_\infty} \xrightarrow{\sim} \mathcal{O}^r_{\mathbb{P}^1_\infty}$$

called a framing at infinity. This is a smooth noncompact complex algebraic variety of complex dimension $2rn$, also known as the instanton moduli space. When $r = 1$, this space is isomorphic to the Hilbert scheme of points in the complex plane, which as a set parametrizes certain ideals in the ring $R = \mathbb{C}[x, y]$,

$$\mathcal{M}_{1,n} \cong \mathrm{Hilb}_n \mathbb{C}^2 = \{I \subset R : \dim_{\mathbb{C}} R/I = n\}. \tag{2}$$

The map is determined by restricting the sheaf to $\mathbb{C}^2 = \mathbb{P}^2 - \mathbb{P}^1_\infty$, and using the framing Φ to obtain the inclusion map to R.

There is a standard torus action on this space

$$G = T^2 \times T^r \curvearrowright \mathcal{M}_{r,n},$$

defined as follows: the action of T^2 is the induced one from

$$T^2 = \mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathbb{P}^2, \quad (z_1, z_2) \cdot (x_0 : x_1 : x_2) = (x_0 : z_1^{-1} x_1 : z_2^{-1} x_2), \quad (3)$$

by pullback of sheaves. The action of T^r is by rotating the framing, i.e.

$$\boldsymbol{w} \cdot \Phi = \operatorname{diag}(\boldsymbol{w}) \cdot \Phi, \quad \boldsymbol{w} = (w_1, ..., w_r) \in T^r,$$

which commutes with the T^2 action.

The fixed points of this action are well known to be isolated, and to correspond to r-tuples of partitions $\boldsymbol{\mu} = (\mu^{(1)}, ..., \mu^{(r)})$ with

$$|\boldsymbol{\mu}| = \sum_i |\mu^{(i)}| = n.$$

Under the isomorphism (2) for $r = 1$, they correspond to the ideals I_μ generated by all monomials $x^i y^j$ such that (i, j) is not a box in the Young diagram of μ, i.e. $j \geqslant \mu_i$. In higher rank $r > 1$, the sheaf and framing $(\mathcal{F}_{\boldsymbol{\mu}}, \varphi_{\boldsymbol{\mu}})$ associated to such a diagram may be determined by the restriction to the plane

$$\mathcal{F}_{\boldsymbol{\mu}}\big|_{\mathbb{C}^2} \cong F_{\boldsymbol{\mu}} := I_{\mu^{(1)}} \oplus \cdots \oplus I_{\mu^{(r)}} \hookrightarrow R^r$$

together with the inclusion map on the right.

There is a bundle $\mathcal{E}$ of rank $r(n_1 + n_2)$ on $\mathcal{M}_{r,n_1} \times \mathcal{M}_{r,n_2}$ whose fiber over a point described by a pair of sheaves $(\mathcal{F}, \mathcal{G})$ is given by

$$\mathcal{E}\big|_{\mathcal{F},\mathcal{G}} = \operatorname{Ext}^1_{\mathbb{P}^2}(\mathcal{F}, \mathcal{G}(-\mathbb{P}^1_\infty)), \quad (4)$$

The restriction of this bundle to the diagonal when $n_1 = n_2$ is well-known to be the tangent bundle. In [10], the K-theory class of the rank one case of this bundle was defined for the Hilbert scheme of points on a general smooth quasi-projective surface. Its Euler class was proved to define an explicit vertex operator in terms of Nakajima's Heisenberg operators, and was further generalized to K-theory in [9]. We have an action

$$G = T^2 \times T^r \times T^r \curvearrowright \mathcal{M}_{r,n_0} \times \mathcal{M}_{r,n_1} \quad (5)$$

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where T^2 acts diagonally, and the second and third tori act on the framings on the first and second components. This action lifts naturally to $\mathcal{E}$ using the description (4), making $\mathcal{E}$ an equivariant bundle.

Now define the torus characters of the fibers of this bundle

$$E_{\boldsymbol{\mu},\boldsymbol{\nu}}(z_1, z_2, \boldsymbol{w}, \boldsymbol{v}) := \operatorname{ch} \mathcal{E}_{\mathcal{F}_{\boldsymbol{\mu}}, \mathcal{F}_{\boldsymbol{\nu}}} \in \mathbb{Z}[z_i^{\pm 1}, w_i^{\pm 1}, v_i^{\pm 1}]$$

where $\boldsymbol{w}, \boldsymbol{v}$ are elements of the first and second r-dimensional torus in (5) respectively. The answer can be expressed in terms of the answer for $r = 1$ by

$$E_{\boldsymbol{\mu},\boldsymbol{\nu}}(z_1, z_2, \boldsymbol{w}, \boldsymbol{v}) = \sum_{i,j} w_i^{-1} v_j E_{\mu^{(i)}, \nu^{(j)}}(z_1, z_2)$$

The rank one case may be calculated by restricting the sheaves to the open subset $\mathbb{C}^2 \subset \mathbb{P}^2$,

$$E_{\mu,\nu}(z_1, z_2) = \chi_R(R, R) - \chi_R(I_\mu, I_\nu),$$

$$\chi_R(F, G) = \sum_{i=0}^{2} (-1)^i \operatorname{ch} \operatorname{Ext}_R^i(F, G) \in \mathbb{Z}((z_1, z_2)) \tag{6}$$

Now by the additivity of the equivariant Euler characteristic on exact sequences, equation (6) may be calculated in any resolution of the modules F, G. We then deduce the explicit formula

$$\chi_R(I_\mu, I_\nu) = \overline{M \operatorname{ch} I_\mu} \operatorname{ch} I_\nu \tag{7}$$

where

$$M = (1 - z_1)(1 - z_2), \quad \operatorname{ch} I_\mu = M^{-1} - Q_\mu, \quad Q_\mu = \sum_{(i,j) \in \mu} z_1^i z_2^j$$

so that $\operatorname{ch} I_\mu$ is simply the torus character of I_μ as a vector space, which lives in $\mathbb{Z}((z_1, z_2))$. The conjugation of one of these power series is determined by simply replacing $z_i = z_i^{-1}$ in its expression as a rational function in z_1, z_2.

For instance, if $\mu, \nu = [1, 1], [2, 1]$, we would find that

$$\operatorname{ch} I_\mu = M^{-1} - 1 - z_2, \quad \operatorname{ch} I_\nu = M^{-1} - 1 - z_1 - z_2,$$

$$\chi_R(I_\mu, I_\nu) = \left(1 - (1 - z_1^{-1})(1 - z_2^{-1})(1 + z_2^{-1})\right) \times$$
$$\left((1 - z_1)^{-1}(1 - z_2)^{-1} - 1 - z_1 - z_2\right),$$

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$$E_{\mu,\nu}(z_1, z_2) = 1 + z_1^{-1}z_2 + z_1 z_2^{-2} + z_1^{-1} + z_2^{-1}. \tag{8}$$

With some work, we can find the explicit combinatorial expression

$$E_{\mu,\nu}(z_1, z_2) = \sum_{s \in \mu} z_1^{-a_\mu(s)-1} z_2^{l_\nu(s)} + \sum_{s \in \nu} z_1^{a_\nu(s)} z_2^{-l_\mu(s)-1}, \tag{9}$$

where $a_\mu(s), l_\mu(s)$ are the arm and leg lengths of the box s in the Young diagram of μ respectively. In the above expression, these lengths may take negative values if s is not inside μ. We may easily verify that this expression matches with example (8). See [10] for details.

We now recall the localization formula. Suppose some variables x_i are indentified as $x_i = \exp(s_i)$, thinking of the x_i as elements of a complex torus $\mathbb{C}^*$, and s_i as elements of its Lie algebra. We define

$$e(\chi) = \prod_i \left(\sum_j a_{ij}s_i \right)^{k_i}, \quad \chi = \sum_i k_i \prod_j x_j^{a_{ij}} \tag{10}$$

which is the same as the equivariant Euler characteristic of a representation with character χ, viewed as an equivariant bundle over a point. For the rest of the paper, we identify the following sets of variables:

$$z_\alpha = e^{t_\alpha}, \quad w_\alpha = e^{a_\alpha}, \quad v_\alpha = e^{b_\alpha}, \quad u_\alpha = e^{m_\alpha}$$

for any subscript α, which might in fact be a pair of indices, i.e. $w_\alpha = w_{ij}$ as we have below. For instance, we would have

$$e\left(z_1 u_1^{-1} - 2w_{12}u_2\right) = \frac{t_1 - m_1}{(a_{12} + m_2)^2}.$$

Let us also set

$$e_m(\chi) = e(e^m \chi) = \prod_i \left(m + \sum_j a_{ij}t_i \right)^{k_i}.$$

The equivariant localization formula for a smooth, projective torus equivariant variety $T^d \circlearrowright X$ with isolated fixed points, and a cohomology class $\gamma \in H_T(X)$, states that

$$\int_X \gamma = \sum_{p \in X^T} \frac{i_p^* \gamma}{e(T_p X)} \tag{11}$$

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where i_p is the inclusion of the fixed point p, and the integral denotes the proper pushforward map to a point, see [1, 13]. If $z_i = e^{t_i} \in T^d$ are the torus variables, then this expression is written as an element of $\mathbb{C}(t_1, ..., t_d)$, but in fact it must reside in $\mathbb{C}[t_1, ..., t_d]$, the equivariant cohomology of a point, implying some cancelation. If X is not compact, then (11) may be taken as a definition, extending integration to a functional satisfying

$$\int_X \gamma = \int_Y \pi_*(\gamma), \quad \gamma : X \to Y$$

for proper maps π. For some cohomology classes γ in a suitable completion of the cohomology ring $H_T(X)$, this definition can be shown to coincide with the usual integration of differential forms vanishing rapidly at infinity even for some noncompact manifolds X.

Now fix a positive integer N, and for any letter x, let $\tilde{x} = (x_1, ..., x_N)$ denote an N-tuple of indexed variables. For a bold letter denoting an r-tuple symbols such as $\boldsymbol{a}$ above, let

$$\tilde{\boldsymbol{a}} = (\boldsymbol{a}_1, ..., \boldsymbol{a}_N), \quad \boldsymbol{a}_i = (a_{i1}, ..., a_{ir})$$

For partitions we will use the superscript, since the subscript is reserved:

$$\tilde{\boldsymbol{\mu}} = \left\{ \boldsymbol{\mu}^{(1)}, ..., \boldsymbol{\mu}^{(N)} \right\}, \quad \boldsymbol{\mu}^{(i)} = (\mu^{(i1)}, ..., \mu^{(ir)})$$

The Nekrasov functions are defined by

$$Z(t_1, t_2, \tilde{\boldsymbol{a}}, \tilde{m}, \tilde{q}) = \sum_{n_1, ..., n_N} q^{n_1 + \cdots + n_N} \int_{\mathcal{M}_{r,n_1} \times \cdots \times \mathcal{M}_{r,n_N}} \prod_{i=1}^{N} \pi_{i,i+1}^*(\mathcal{E}) =$$

$$\sum_{\tilde{\boldsymbol{\mu}}} \prod_{i=1}^{N} q_i^{|\boldsymbol{\mu}^{(i)}|} \frac{w_{\boldsymbol{\mu}^{(i)}, \boldsymbol{\mu}^{(i+1)}}(t_1, t_2, \boldsymbol{a}_i, \boldsymbol{a}_{i+1}, m_i)}{w_{\boldsymbol{\mu}^{(i)}, \boldsymbol{\mu}^{(i)}}(t_1, t_2, \boldsymbol{a}_i, \boldsymbol{a}_i, 0)}, \tag{12}$$

where $\pi_{i,j}$ is the projection onto the ith and jth factor,

$$w_{\boldsymbol{\mu}, \boldsymbol{\nu}}(t_1, t_2, \boldsymbol{a}_i, \boldsymbol{a}_j, m) = e_m \left(E_{\boldsymbol{\mu}, \boldsymbol{\nu}}(z_1, z_2, \boldsymbol{w}_i, \boldsymbol{w}_j) \right),$$

and we identify $N + 1$ with 1. The sum is over all N-tuples of r-tuples of partitions, not just those of a fixed norm. For $N = 1$, this is the "instanton part" of the partition functions originally written down by Nekrasov in [24] in the form of contour integrals. For a mathematical introduction to the

meaning of these integrals, we refer the reader to Okounkov's ICM notes [26].

For example, for $r = 2, N = 1$ we would have

$$Z(t_1, t_2, (a, -a), m, q) =$$

$$1 + \left(\frac{(m - 2a)(m - t_2)(m - t_1)(m + 2a - t_1 - t_2)}{2at_1t_2(t_1 + t_2 - 2a)} - \right.$$
$$\left. \frac{(m + 2a)(m - t_2)(m - t_1)(m - 2a - t_1 - t_2)}{2at_1t_2(t_1 + t_2 + 2a)} \right) q + O(q^2) \qquad (13)$$

up to first order in q.

2.2   Conformal Blocks

The second side of the AGT relations are formed by the conformal blocks of the Virasoro algebra. See [21] for many useful calculations in the context of AGT.

Recall the commutation relations for the Virasoro algebra given by

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,n} \frac{m^3 - m}{12} K, \qquad (14)$$

where K is central. Let $M_h = M_{h,c}$ denote the Verma module of the Virasoro algebra of level h and central charge c, so that the Cartan subalgebra generated by L_0, K acts on its lowest weight (vaccuum) vector $v_\varnothing$ by

$$X v_\varnothing = \Lambda_{h,c}(X) v_\varnothing, \quad \Lambda_{h,c}(L_0) = h, \quad \Lambda_{h,c}(K) = c,$$

whereas positive generators L_k for $k > 0$ annihilate it. There is a basis of M_h indexed by partitions given by

$$v_\mu = L_{-\mu} v_\varnothing, \quad L_{-\mu} = L_{-\mu_1} \cdots L_{-\mu_l}.$$

We will also write $v_{\mu,h}$ if we want to stress that $v_{\mu,h}$ is an element of M_h.

There an inner product on M_h called the Shapovalov form defined by

$$(v_\varnothing, v_\varnothing)_h = 1, \quad (L_{-n}v_\mu, v_\nu)_h = (v_\mu, L_n v_\nu)_h.$$

We denote its matrix elements by

$$K_{\mu,\nu}(h) = (v_\mu, v_\nu)_h$$

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For instance, the entries $K_{\mu,\nu}(h)$ for $\mu, \nu \in \{[2],[1,1]\}$ are given by

$$\begin{pmatrix} 4h + c/2 & 6h \\ 6h & 8h^2 + 4h \end{pmatrix}.$$

There is an intertwining operator called the *Liouville vertex operator*

$$\mathcal{V}^h_{k_2,k_1}(x) : M_{k_1} \to M_{k_2}$$

satisfying

$$\left(\mathcal{V}^h_{k_2,k_1}(x) v_{\varnothing,k_1}, v_{\varnothing,k_2} \right)_{k_2} = x^{k_2 - h - k_1}, \tag{15}$$

$$[L_n, \mathcal{V}^h_{k_2,k_1}(x)] = \left(h(n+1)x^n + x^{n+1}\partial_x \right) \mathcal{V}^h_{k_2,k_1}(x). \tag{16}$$

We will find it more convenient to use the normalization

$$V^h_{k_2,k_1}(x) = x^{k_1 + h - k_2} \mathcal{V}^h_{k_2,k_1}(x),$$

so that the vaccum matrix element is one. Strictly speaking, this operator is only defined as a *field*, i.e. a formal power series

$$\mathcal{V}^h_{k_2,k_1}(x) \in \mathrm{Hom}(M_{k_1}, M_{k_2})[[x^{\pm 1}]],$$

with the property that each matrix element is a formal Laurent series, see [5]. However, the full Liouville vertex operator may be defined to act on a larger Hilbert space of functions on a noncompact space, see [30] for an introduction.

We set

$$S_{\mu,\nu}(k_1, h, k_2) = S_{k_1,h,k_2}(v_\mu, v_\nu) = \left(\mathcal{V}^h_{k_2,k_1}(x) v_{\mu,k_1}, v_{\nu,k_2} \right)_{k_2} \Big|_{x=1}.$$

For instance, we have

$$S_{[1],[1]}(k_1, h, k_2) = S_{k_1,h,k_2}(L_{-1}v_\varnothing, L_{-1}v_\varnothing) =$$

$$S_{k_1,h,k_2}(v_\varnothing, L_1 L_{-1}v_\varnothing) + (k_1 + h - k_2 - 1)S(v_\varnothing, L_{-1}v_\varnothing) =$$

$$2k_2 + (k_1 + h - k_2 - 1)(k_2 + h - k_1). \tag{17}$$

More complicated terms will of course involve the central charge c.

Let $d = L_0 - h$ so that $dv_\mu = |\mu|v_\mu$. Let

$$\mathcal{B}(c, \tilde{h}, \tilde{k}, \tilde{q}) = \mathrm{Tr}\, q^d V^{h_N}_{k_1,k_N}(x_N) \cdots V^{h_2}_{k_3,k_2}(x_2) V^{h_1}_{k_2,k_1}(x_1) =$$

$$\sum_{\tilde{\mu},\tilde{\nu}} q_1^{|\mu^{(1)}|} \cdots q_N^{|\mu^{(N)}|} \frac{S_{\nu_1,\mu_2}(k_1, h_1, k_2) \cdots S_{\nu_N,\mu_1}(k_N, h_N, k_1)}{K_{\mu_1,\nu_1}(k_1) \cdots K_{\mu_N,\nu_N}(k_N)}$$

where

$$q = x_1 = q_1 \cdots q_N, \qquad x_i x_{i+1}^{-1} = q_{i+1} \tag{18}$$

2.3 The AGT conjecture

The AGT relations for a genus 1 Riemann surface with N punctures state that

Conjecture 1. *(AGT [3]) Let $\boldsymbol{a}_i = (a_i, -a_i)$ for $1 \leqslant i \leqslant N$. Then we have that*

$$Z(t_1, t_2, \tilde{\boldsymbol{a}}, \tilde{m}, \tilde{q}) = Z'(t_1, t_2, \tilde{m}, \tilde{q}) \mathcal{B}(c, \tilde{k}, \tilde{h}, \tilde{q}) \tag{19}$$

under the substitution

$$c = 1 + 6\frac{(t_1 + t_2)^2}{t_1 t_2}, \quad k_i = \frac{(t_1 + t_2)^2 - 4a_i^2}{4t_1 t_2},$$

$$h_i = \Delta_{m_i, m_i}, \quad \Delta_{m,n} = \frac{m(t_1 + t_2 - n)}{t_1 t_2}, \tag{20}$$

and

$$Z'(t_1, t_2, \tilde{m}, \tilde{q}) = (q; q)_\infty^{2\Delta_{m_1, m_1} + \cdots + 2\Delta_{m_N, m_N} - 1} \times$$

$$\prod_{i<j} (x_i x_j^{-1}; q)_\infty^{2\Delta_{m_i, m_j}} (x_i^{-1} x_j q; q)_\infty^{2\Delta_{m_j, m_i}}$$

where the x_i are determined by (18), and $(x; q)_\infty = \prod_{i \geqslant 0}(1 - xq^i)$.

For instance, for $N = 1$ we would have

$$Z'(t_1, t_2, m, q) = (q; q)_\infty^{2\Delta_{m,m} - 1} = 1 + \left(1 - \frac{2m(t_1 + t_2 - m)}{t_1 t_2}\right) q + \cdots$$

and

$$\mathcal{B}(c, k, h) = 1 + S_{[1],[1]}(k, h, k) K_{[1],[1]}^{-1}(k) q + \cdots =$$

$$1 + \frac{h^2 - h + 2k}{2k} q + \cdots$$

using our example (17) at $k_1, k_2 = k$. Using (13), we can check that

$$Z(t_1, t_2, (a, -a), m, q) = Z'(t_1, t_2, m, q) \mathcal{B}(c, k, h, q)$$

to first order in q, after the substitution

$$k = \frac{(t_1 + t_2)^2 - 4a}{t_1 t_2}, \quad h = \frac{m(t_1 + t_2 - m)}{t_1 t_2}.$$

The central charge c does not appear until higher order.

3 Vertex operators

3.1 The infinite wedge representation

Let $\Lambda = \Lambda^{\infty/2}$ denote the infinite wedge representation, which we will now briefly summarize and refer to [6, 17] for details. It has a basis labeled by partitions which we write as

$$v_\mu = v_{\mu_1} \wedge v_{\mu_2-1} \wedge v_{\mu_3-2} \wedge \cdots$$

The possible sequences

$$(i_1, i_2, i_3, ...) = (\mu_1, \mu_2 - 1, \mu_3 - 2, ...)$$

that can appear are precisely the strictly decreasing sequences of integers such that the number of entries with $i_k > 0$ equals the number of $i_k \leqslant 0$. This space can be thought of as the $\infty/2$ exterior power of the vector space $\mathbb{C} \cdot \mathbb{Z}$ with basis v_i indexed by the integers. The wedge product of sums of basis vectors v_i should be distributed and sorted with signs in the usual way. For instance,

$$v_2 \wedge (v_1 + v_4) \wedge v_{-2} \wedge \cdots =$$

$$(v_2 \wedge v_1 \wedge v_{-2} \wedge \cdots) - (v_4 \wedge v_2 \wedge v_{-2} \wedge \cdots) = v_{[2,2]} - v_{[4,3]}.$$

There is an action of the Lie algebra of the infinite-dimensional general linear group $\mathfrak{gl}(\mathbb{C} \cdot \mathbb{Z})$ defined by

$$\rho'(E_{ij}) \cdot v_{i_1} \wedge v_{i_2} \wedge \cdots =$$

$$\sum_k v_{i_1} \wedge \cdots \wedge v_{i_{k-1}} \wedge E_{ij} v_{i_k} \wedge v_{i_{k+1}} \wedge \cdots$$

where

$$E_{ij} v_k = \begin{cases} v_i & j = k \\ 0 & \text{otherwise} \end{cases}$$

This extends to a projective representation defined by

$$\rho(E_{ij}) = \begin{cases} \rho'(E_{ij}) & \text{if } i \neq j \text{ or } i = j > 0 \\ \rho'(E_{ij}) - Id & \text{if } i = j \leqslant 0 \end{cases} \tag{21}$$

which has the commutation relations

$$[\rho(E_{ij}), \rho(E_{kl})] = \rho([E_{ij}, E_{kl}]) + \epsilon_{ijkl} Id,$$

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$$\epsilon_{ijkl} = \begin{cases} 1 & i = l \leqslant 0 \text{ and } j = k \geqslant 1 \\ -1 & i = l \geqslant 1 \text{ and } j - k \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

The projective representation extends to some infinite sums of the elementary matrices E_{ij} which are useful for defining the action of Kaç-Moody algebras on Λ. For instance, the element

$$d = \sum_{i \in \mathbb{Z}} i\rho\left(E_{ii}\right)$$

becomes a finite sum when applied to any vector, and is determined by

$$d \cdot v_\mu = |\mu|v_\mu.$$

We also define the action of the infinite-dimensional Heisenberg Lie algebra

$$\alpha_n = \sum_{i \in \mathbb{Z}} \rho\left(E_{i,i+n}\right)$$

satisfying

$$[\alpha_m, \alpha_n] = m\delta_{m,n}Id. \tag{23}$$

There is an isomorphism from the polynomial algebra

$$\mathbb{C}[\alpha_{-1}, \alpha_{-2}, ...] \cong \Lambda,$$

given by simply applying the polynomial on the left to the vacuum $v_\varnothing$, so the image of the monomials would be

$$\alpha_\lambda = \alpha_{-\lambda_1} \cdots \alpha_{-\lambda_l} \cdot v_\varnothing$$

for a Young diagram λ of length l. To determine the inverse map amounts to finding coefficients in the expansion

$$v_\mu = \sum_\lambda c_{\lambda,\mu}\alpha_\lambda.$$

It turns out that the $c_{\lambda,\mu}$ are precisely the coefficients of the expansion of the Schur polynomial s_μ in the power sum basis p_λ [18].

Now for each m, we have the following well-known *vertex operator*,

$$\Gamma^{(m)}(x) = \Gamma_-^m(x)\Gamma_+^{-m}(x^{-1}) \in \text{End}\left(\Lambda\right)\left[\left[x^{\pm 1}\right]\right], \tag{24}$$

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where

$$\Gamma_\pm^m(x) = \exp\left(m \sum_{k>0} \frac{x^k \alpha_{\pm k}}{k}\right).$$

We will also use the odd and even parts, given by

$$\Gamma_s^{(m)}(x) = \Gamma_{s,-}^m(x)\Gamma_{s,+}^{-m}(x^{-1})$$

where $s = e, o$, and

$$\Gamma_{e,\pm}(x) = \exp\left(m \sum_{k>0,even} \frac{x^k \alpha_{\pm k}}{k}\right), \quad \Gamma_{o,\pm}(x) = \exp\left(m \sum_{k>0,odd} \frac{x^k \alpha_{\pm k}}{k}\right)$$

so that $\Gamma^{(m)}(x) = \Gamma_e^{(m)}(x)\Gamma_o^{(m)}(x)$. We will also write

$$\gamma\Gamma^{(m)}(x)\gamma^{-1} = \Gamma_e^{(m)}(x) \otimes \Gamma_o^{(m)}(x) \tag{25}$$

where

$$\gamma : \Lambda \to \mathbb{C}[\alpha_{-1}, \alpha_{-2}, ...] \cong$$

$$\mathbb{C}[\alpha_{-2}, \alpha_{-4}, ...] \otimes \mathbb{C}[\alpha_{-1}, \alpha_{-3}, ...] =: \Lambda_e \otimes \Lambda_o.$$

By exponentiating the commutation relations (23), we can determine that

$$\Gamma_{s,+}^m(x)\Gamma_{s,-}^n(y) = \Omega_s(x,y)^{mn}\Gamma_{s,-}^n(y)\Gamma_{s,+}^m(x), \tag{26}$$

where

$$\Omega_e(x,y) = \frac{1}{\sqrt{1-x^2y^2}}, \quad \Omega_o(x,y) = \frac{\sqrt{1-x^2y^2}}{(1-xy)}$$

so that

$$\Omega_e(x,y)\Omega_o(x,y) = \Omega(x,y) = \frac{1}{1-xy}.$$

We also have the easier relation

$$q^d\Gamma_{s,\pm}(x) = \Gamma_{s,\pm}(xq^{\mp 1})q^d. \tag{27}$$

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3.2 The principal vertex operator construction

Now let $\hat{\mathfrak{g}}$ denote the affine Lie algebra $\hat{\mathfrak{sl}}_2\mathbb{C}$, whose underlying vector space is given by

$$\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_2\mathbb{C} = \mathfrak{sl}_2(\mathbb{C}[t, t^{-1}]) + \mathbb{C}d' + \mathbb{C}K.$$

Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

denote the standard generators of $\mathfrak{sl}_2\mathbb{C}$. For each $i \in \mathbb{Z}$ and $a \in \mathfrak{sl}_2\mathbb{C}$, let

$$a(x) = \sum_{k \in \mathbb{Z}} a_{-k}x^k \in \hat{\mathfrak{g}}[[x^{\pm 1}]], \quad a_k = a \cdot t^{-k} \in \hat{\mathfrak{g}}$$

The commutators are given by

$$[a_m, b_n] = [a, b]_{m+n} + m\delta_{m,-n}K,$$

$$[d', a_k] = ka_{k-1}, \quad [K, \hat{\mathfrak{g}}] = 0. \tag{28}$$

There is an action of $\hat{\mathfrak{g}}$ on Λ which is induced from the action

$$\mathfrak{sl}_2\left(\mathbb{C}[t, t^{-1}]\right) \curvearrowright \mathbb{C}^2[t, t^{-1}] \cong \mathbb{C} \cdot \mathbb{Z}$$

where the isomorphism is defined by

$$(t^k, 0) \mapsto v_{2k}, \quad (0, t^k) \mapsto v_{2k-1}.$$

Explicitly, we have

$$e_i \mapsto \sum_k E_{2k, -1+2i+2k}, \quad h_i \mapsto \sum_k E_{2k, 2i+2k} - E_{-1+2k, -1+2i+2k},$$

$$f_i \mapsto \sum_k E_{-1+2k, 2i+2k}, \quad 2d' + \frac{1}{2}h_0 \mapsto d, \quad K \mapsto 1, \tag{29}$$

where it is understood the the elementary matrices E_{ij} act via the projective representation ρ. This representation is not irreducible, but the span of $\hat{\mathfrak{g}}$ applied to $v_{\varnothing}$ is isomorphic to the basic representation Λ_0.

The principal vertex operator construction gives an additional description of the action (29), which explicitly identifies it as the space $\Lambda_o \subset \Lambda$ defined above. It is described by

$$2d' \mapsto d - A_0/2, \quad K \mapsto 1,$$

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$$h_i \mapsto A_{2i}, \quad 2e_i \mapsto \alpha_{2i-1} - A_{2i-1}, \quad 2f_i \mapsto \alpha_{2i+1} + A_{2i+1} \qquad (30)$$

where

$$2A(x) = 2\sum_i A_{-i} x^i = \Gamma_o^{(2)}(x) - 1.$$

3.3 The Segal-Sugawara construction

We now recall some facts about the Segal-Sugawara construction, for which we refer to [5]. Let V be a representation of an affine Lie algebra $\hat{\mathfrak{g}}$ with central charge c. The Segal-Sugawara construction produces operators L_n in terms of the generators of $\hat{\mathfrak{g}}$ such that

1. The vector space V becomes a representation of the Virasoro algebra with central charge
$$c' = \frac{c \dim \mathfrak{g}}{c + h^\vee},$$
where $h^\vee$ is the dual Coxeter number. In particular, if $\mathfrak{g} = \mathfrak{sl}_2$ and $V = \Lambda_0$, we would have $c = 1, h^\vee = 2, \dim \mathfrak{g} = 3$, so that $c' = 1$.

2. The Virasoro algebra interwines the action of $\hat{\mathfrak{g}}$ in the desired manner, coming from the action of automorphism of the circle on $Map(S^1, \mathfrak{g})$ by precomposition, i.e.
$$[L_m, a(x)] = x^{1+m} \partial_x a(x) \quad a \in \mathfrak{g}.$$

Furthermore, L_0 coincides with the differential d'.

In the case $\mathfrak{g} = \mathfrak{sl}_2 \mathbb{C}$, the construction has the form

$$L_k = \frac{1}{12} \sum_{i \in \mathbb{Z}} : 2e_i f_{k-i} + 2f_i e_{k-i} + h_i h_{k-i} : \qquad (31)$$

where the "normal ordering" symbol means

$$: a_i b_j := \begin{cases} a_i b_j & \text{if } i \leqslant 0 \\ b_j a_i & \text{otherwise.} \end{cases}$$

Combining this formula with (29), we arrive at an action of the Virasoro algebra with central charge 1 on Λ which preserves Λ_o. In fact, there is a family of actions of $\mathcal{V}ir$ parametrized by a number s given by

$$L_{k,s} = L_k + s h_k + s^2 \delta_{k,0}. \qquad (32)$$

It is straightforward to verify that these also satisfy (14). For integer values of s these are the translates of the original action by the translation subgroup of the affine Weyl group of $\hat{\mathfrak{sl}}_2$.

We have a decomposition of Λ_o as a representation over $L_{k,s}$ for any s as follows. By the Kaç character formula, we have that

$$\mathrm{Tr}_{\Lambda_o} y^{h_0} q^{d'} = \sum_{k \in \mathbb{Z}} y^{2k} q^{k^2} (q;q)_\infty^{-1}, \quad (x;q)_\infty = \prod_{i \geq 0}(1 - xq^i). \tag{33}$$

Since the Virasoro action commutes with h_0, we find that Λ_o decomposes as

$$\Lambda_o \cong \bigoplus_k V_k$$

where V_k is the eigenspace of h_0 with eigenvalue $2k$, and contains a unique up to scalar lowest eigenvector v_k of d' with eigenvalue k^2, given by $v_k = v_\mu$ where

$$\mu = \begin{cases} [2k, 2k-1, ..., 1] & k \geq 0 \\ [-2k-1, -2k-2, ..., 1] & k < 0 \end{cases} \tag{34}$$

We find that

$$L_{0,s} v_k = (d' + sh_0 + s^2)v_k = (k+s)^2 v_k,$$

giving rise to a map

$$M_{(k+s)^2} \to V_k. \tag{35}$$

Furthermore, if s is in the range where $M_{(k+s)^2}$ is irreducible, then this map is injective. Using (33), we find that

$$\mathrm{Tr}_{M_{(k+s)^2}} q^{L_{0,s}} = \mathrm{Tr}_{V_k} q^{L_{0,s}} = q^{(k+s)^2}(q;q)_\infty^{-1}$$

so the map is also an isomorphism in that range.

The following proposition will be used to prove our main theorem:

Proposition 1. *We have that*

$$[L_{k,1/4}, \Gamma_o^{(2m)}(x^{1/2})] = \left(m^2 k x^k + x^{k+1}\partial_x\right)\Gamma_o^{(2m)}(x^{1/2}) \tag{36}$$

Proof. We begin with the case $m = 1$. In this case, using (30), we have that

$$\Gamma_o^{(2)}(x^{1/2}) = -2x^{1/2}e(x) + 2h(x) + 2x^{-1/2}f(x) + 1 =$$

$$x^{h_0/4} a(x) x^{-h_0/4} + 1, \quad a = -2f + 2h + 2f.$$

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Then we have

$$[L_{k,1/4}, \Gamma_o^{(2)}(x^{1/2})] = [L_k + h_k/4 + 1/16, x^{h_0/4}a(x)x^{-h_0/4} + 1] =$$

$$x^{h_0/4}\left(kx^k + x^{k+1}\partial_x\right)a(x)x^{-h_0/4}+$$

$$1/4 \cdot \left(x^k x^{h_0/4}[h,a](x)x^{-h_0/4} + 4kx^k\right) =$$

$$kx^k x^{h_0/4}a(x)x^{-h_0/4} + x^{k+1}\partial_x\left(x^{h_0/4}a(x)x^{-h_0/4}\right) + kx^k =$$

$$\left(kx^k + x^{k+1}\partial_x\right)\Gamma_o^{(2)}(x^{1/2}).$$

We now suppose now that (36) holds for some m, n, and prove that it holds for $m + n$. Using (26), we find that

$$\Gamma_o^{(2(m+n))}(x^{1/2}) = \lim_{y \to x} \Gamma_{-,o}^{2m}(x^{1/2})\Gamma_{-,o}^{2n}(y^{1/2})\Gamma_{+,o}^{-2m}(x^{-1/2})\Gamma_{+,o}^{-2n}(y^{-1/2}) =$$

$$\lim_{y \to x} A(x,y)\Gamma_o^{(2m)}(x^{1/2})\Gamma_o^{(2n)}(y^{1/2}), \quad A(x,y) = \Omega_o(x^{-1/2}, y^{1/2})^{4mn}$$

We then have

$$[L_{k,1/4}, \Gamma_o^{(2m+2n)}(x^{1/2})] =$$

$$\lim_{y \to x} A(x,y)(km^2x^k + kn^2y^k + x^{k+1}\partial_x + y^{k+1}\partial_y)$$

$$\Gamma_o^{(2m)}(x^{1/2})\Gamma_o^{(2n)}(y^{1/2}) =$$

$$\lim_{y \to x} A(x,y)(km^2x^k + kn^2y^k + x^{k+1}\partial_x + y^{k+1}\partial_y)$$

$$A(x,y)^{-1}\Gamma_{-,o}^{2m}(x^{1/2})\Gamma_{-,o}^{2n}(y^{1/2})\Gamma_{+,o}^{-2m}(x^{-1/2})\Gamma_{+,o}^{-2n}(y^{-1/2}) =$$

$$(k(m^2 + n^2)x^k + x^{k+1}\partial_x)\Gamma_o^{(2m+2n)}(x^{1/2})+$$

$$\left(\lim_{y \to x} A(x,y)(x^{k+1}\partial_x + y^{k+1}\partial_y)A(x,y)^{-1}\right)\Gamma_o^{(2m+2n)}(x^{1/2}).$$

The limit in parentheses in the last line can be seen to equal $2kmnx^k$, which combines with the first expression to establish (36).

□

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4 Application to AGT

4.1 Nekrasov functions for $t_1 + t_2 = 0$

We now explain how the infinite wedge representation was applied to Nekrasov functions in [25].

First, we restrict our torus action by specializing

$$z_1 = z, \quad z_2 = z^{-1}, \quad t_1 = t, \quad t_2 = -t, \quad z = e^t \tag{37}$$

Now substitute (37) into (7) to get

$$E_{\boldsymbol{\mu},\boldsymbol{\nu}}(z,\boldsymbol{w},\boldsymbol{v}) = E_{\boldsymbol{\mu},\boldsymbol{\nu}}(z,z^{-1},\boldsymbol{w},\boldsymbol{v}) = \sum_{i,j=1}^{r} w_i^{-1} v_j E_{\mu,\nu}(z),$$

$$E_{\mu,\nu}(z) = \chi_{\varnothing,\varnothing}(z) - \chi_{\mu,\nu}(z), \quad \chi_{\mu,\nu}(z) = f_\mu(z^{-1}) f_\nu(z) \tag{38}$$

where $f_\mu(z) \in \mathbb{C}(z)$ is the rational function whose Laurent series about infinity is given by

$$f_\mu(z) = \sum_{i \geqslant 1} z^{\mu_i - i + 1} = \sum_{i=1}^{\ell(\mu)} z^{\mu_i - i + 1} + \frac{z^{-\ell(\mu)-1}}{(1 - z^{-1})}.$$

Notice that the expression in (38) must be a polynomial, implying some cancellation.

For the rest of the paper we will assume without any additional loss of information that $t = 1$ and write

$$Z(\tilde{\boldsymbol{a}}, \tilde{m}, \tilde{q}) = Z(1, -1, \tilde{\boldsymbol{a}}, \tilde{m}, \tilde{q}),$$

$$Z'(\tilde{m}, \tilde{q}) = Z'(1, -1, \tilde{m}, \tilde{q}), \quad \mathcal{B}(\tilde{h}, \tilde{k}) = \mathcal{B}(1, \tilde{h}, \tilde{k}).$$

Set

$$w_{\boldsymbol{\mu},\boldsymbol{\nu}}(\boldsymbol{a}, \boldsymbol{b}, m) = e_m \left(E_{\boldsymbol{\mu},\boldsymbol{\nu}}(z) \right), \quad w_{\mu,\nu}(m) = e_m \left(E_{\mu,\nu}(z) \right) \tag{39}$$

By the symmetry $z \leftrightarrow z^{-1}$, there is a polynomial $w_{\boldsymbol{\mu}}(\boldsymbol{a})$ such that

$$w_{\boldsymbol{\mu}}(\boldsymbol{a}) := w_{\boldsymbol{\mu},\boldsymbol{\mu}}(\boldsymbol{a}, \boldsymbol{a}, 0) = (-1)^{r|\boldsymbol{\mu}|} w_{\boldsymbol{\mu}}(\boldsymbol{a})^2 \tag{40}$$

If $r = 1$, the w_μ is simply the product of the hook lengths of μ.

For instance, we would have

$$f_{[2,1]}(z) = z^2 + 1 + \frac{z^{-2}}{1 - z^{-1}}, \quad f_{[1,1]}(z) = z + 1 + \frac{z^{-2}}{1 - z^{-1}}.$$

This would give

$$E_{[1,1],[2,1]}(z) = \frac{1}{(1-z)(1-z^{-1})} - f_{[1,1]}(z^{-1})f_{[2,1]}(z) =$$

$$z^3 + z + 1 + z^{-1} + z^{-2},$$

which agrees with (8) under the specialization (37). We then have

$$w_{[1,1],[2,1]}(m) = (m+3)(m+1)m(m-1)(m-2). \tag{41}$$

Now let $\mathcal{H}^r$ denote the complex vector space with basis vectors given by $u_{\boldsymbol{\mu}}$, where $\boldsymbol{\mu}$ is an r-tuple of partitions, and let $\mathcal{H} = \mathcal{H}^1$. We define an operator

$$W_{\boldsymbol{a},\boldsymbol{b}}^{(m)}(x) \in \mathrm{End}(\mathcal{H}^r)[[x^{\pm 1}]],$$

$$\left(W_{\boldsymbol{b},\boldsymbol{a}}^{(m)}(x)u_{\boldsymbol{\mu}}, u_{\boldsymbol{\nu}}\right) = (-1)^{r|\boldsymbol{\mu}|}x^{|\boldsymbol{\nu}|-|\boldsymbol{\mu}|}\frac{w_{\boldsymbol{\mu},\boldsymbol{\nu}}(\boldsymbol{a},\boldsymbol{b},m)}{w_{\boldsymbol{\mu}}(\boldsymbol{a})w_{\boldsymbol{\nu}}(\boldsymbol{b})}. \tag{42}$$

If $r = 1$ we will simply write $W^{(m)}(x)$. We then have

$$Z(\tilde{\boldsymbol{a}}, \tilde{m}, \tilde{q}) = \mathrm{Tr}_{\mathcal{H}^r}\, q^d W_{\boldsymbol{a}_1, \boldsymbol{a}_N}^{(m_N)}(x_N) \cdots W_{\boldsymbol{a}_3, \boldsymbol{a}_2}^{(m_2)}(x_2) W_{\boldsymbol{a}_2, \boldsymbol{a}_1}^{(m_1)}(x_1) \tag{43}$$

where the x_i are related to the q_i by (18).

Consider the following isomorphism:

$$\iota : \mathcal{H} \to \Lambda, \quad u_\mu \mapsto v_\mu$$

The following result was proved in [25], and was extended to the general action (3) (and in fact, to a general smooth quasiprojective surface) by Okounkov and the author in [10]. A further extension of this theorem to K-theory may be found in [9]. The author has also described a very short proof for the more general K-theoretic version, but for the specialized action (37) in [8].

Proposition 2. *We have that*

$$\iota W^{(m)}(x)\iota^{-1} = \Gamma^{(m)}(x). \tag{44}$$

For instance, dropping the power of x, we find that

$$(\Gamma^{(m)}v_{[1,1]}, v_{[2,1]}) = (\Gamma_+^{-m}v_{[1,1]}, \Gamma_+^{m}v_{[2,1]}).$$

Next, we get

$$\Gamma_+^{-m}v_{[1,1]} = \Gamma_+^{-m}v_1 \wedge \Gamma_+^{-m}v_0 \wedge \Gamma_+^{-m}v_{-2} \wedge \cdots =$$

$$\left(v_1 - mv_0 + \frac{m(m-1)}{2}v_{-1} + \cdots\right) \wedge$$

$$(v_0 - mv_{-1} + \cdots) \wedge (v_{-2} + \cdots) \wedge \cdots =$$

$$v_{[1,1]} - mv_{[1]} + \frac{m^2 + m}{2}v_{\varnothing},$$

and

$$\Gamma_+^{m}v_{[2,1]} = \Gamma_+^{m}v_2 \wedge \Gamma_+^{m}v_0 \wedge \Gamma_+^{m}v_{-2} \wedge \cdots =$$

$$\left(v_2 + mv_1 + \frac{m(m+1)}{2}v_0 + \frac{m(m+1)(m+2)}{6}v_{-1} + \cdots\right) \wedge$$

$$(v_0 + mv_{-1} + \cdots) \wedge (v_{-2} + \cdots) \wedge \cdots =$$

$$v_{[2,1]} + mv_{[2]} + mv_{[1,1]} + m^2 v_{[1]} + \frac{m^3 - m}{3}v_{\varnothing}.$$

Taking the inner product of the two yields

$$\frac{m(m-1)(m-2)(m+3)(m+1)}{6}$$

which agrees with (41), (42), and (44).

We now explain how to apply proposition 2 to higher rank, which was used by Okounkov and Nekrasov to compute the *dual partition function* in [25]. Consider the function

$$\mathcal{B}_r : \mathcal{P}^r \times \mathbb{Z}^r \to \mathcal{P} \times \mathbb{Z}$$

which associates to an r-tuple a *blended partition*

$$(\boldsymbol{\mu}, \boldsymbol{k}) = (\mu^{(1)}, ..., \mu^{(r)}; k_1, ..., k_r) \mapsto (\mu, k),$$

where μ, k are the uniquely determined by the property that

$$\{\mu_i - i + 1 + k\}_{i \geqslant 1} = \bigcup_{j=1}^{r}\left\{r\left(\mu_i^{(j)} - i + 1 + k_j\right) - j + 1\right\}_{i \geqslant 1} \qquad (45)$$

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and $k = |\boldsymbol{k}| = k_1 + \cdots + k_r$. The norms are related by

$$|\mu| = r|\boldsymbol{\mu}| + d_{\boldsymbol{k}}, \quad d_{\boldsymbol{k}} = \frac{r-1}{2}\sum_i k_i^2 + \frac{r+1-2i}{2}k_i - \sum_{i<j} k_i k_j. \qquad (46)$$

It is straightforward to see that this map is bijective. If we have $|\boldsymbol{k}| = 0$ then we obtain another isomorphism

$$\beta_{\boldsymbol{k}} : \mathcal{H}^r \to \mathcal{H}, \quad u_{\boldsymbol{\mu}} \mapsto u_\mu, \quad \mathcal{B}_r(\boldsymbol{\mu}, \boldsymbol{k}) = (\mu, 0)$$

Proposition 3. *We have*

$$\beta_{\boldsymbol{l}}^{-1} W^{(rm)}(x) \beta_{\boldsymbol{k}} = c_{\boldsymbol{k},\boldsymbol{l},m} x^{d_{\boldsymbol{l}} - d_{\boldsymbol{k}}} W^{(m)}_{\boldsymbol{l}-\boldsymbol{i}/r, \boldsymbol{k}-\boldsymbol{i}/r}(x^r) \qquad (47)$$

where $\boldsymbol{i} = (1, 2, ..., r)$, and $c_{\boldsymbol{k},\boldsymbol{l},m}$ is a constant.

Proof. If

$$(\mu, 0) = \mathcal{B}_r(\boldsymbol{\mu}, \boldsymbol{k}), \quad (\nu, 0) = \mathcal{B}_r(\boldsymbol{\mu}, \boldsymbol{l}),$$

then we have

$$f_\mu(z) = \sum_{j=1}^r z^{k_j - j + 1} f_{\mu^{(j)}}(z^r),$$

and therefore

$$E_{\mu,\nu}(z) - E_{\mu^0,\nu^0}(z) = E_{\boldsymbol{\mu},\boldsymbol{\nu}}(z^r, z^{r\boldsymbol{k}-\boldsymbol{i}}, z^{r\boldsymbol{l}-\boldsymbol{i}})$$

where

$$(\mu^0, 0) = \mathcal{B}_r(\varnothing, \boldsymbol{k}), \quad (\nu^0, 0) = \mathcal{B}_r(\varnothing, \boldsymbol{l})$$

are the blended r-tuples of empty partitions.

It follows that

$$w_{\mu,\nu}(rm) \sim e_{rm}\left(E_{\boldsymbol{\mu},\boldsymbol{\nu}}(z^r, z^{r\boldsymbol{k}-\boldsymbol{i}}, z^{r\boldsymbol{l}-\boldsymbol{i}})\right) =$$

$$r^{2r(|\boldsymbol{\mu}|+|\boldsymbol{\nu}|)} w_{\boldsymbol{\mu},\boldsymbol{\nu}}(\boldsymbol{k} - \boldsymbol{i}/r, \boldsymbol{l} - \boldsymbol{i}/r, m)$$

where $\sim$ means the left side is a multiple of the right side by a constant that does not depend on $\boldsymbol{\mu}, \boldsymbol{\nu}$. This also implies that

$$w_\mu(rm) \sim r^{2r|\boldsymbol{\mu}|} w_{\boldsymbol{\mu}}(\boldsymbol{k} - \boldsymbol{i}/r, \boldsymbol{l} - \boldsymbol{i}/r, m)$$

Inserting both of these and (46) into (42) yields the result up to the sign, which is straightforward to determine. $\qquad \square$

4.2 The main theorem

We may now state our main theorem.

Theorem 1. *Let $\boldsymbol{k} = (k, -k), \boldsymbol{l} = (l, -l)$, and consider the composition*

$$\gamma_{\boldsymbol{k}} : \mathcal{H}^2 \xrightarrow{\beta_{\boldsymbol{k}}} \mathcal{H} \xrightarrow{\iota} \Lambda \xrightarrow{\gamma} \Lambda_e \otimes \Lambda_o.$$

Then

a) The map $\gamma_{\boldsymbol{k}}$ is injective, and its image is $\Lambda_e \otimes V_k$. We may therefore write

$$\gamma_{\boldsymbol{k}} : \mathcal{H}^2 \to \Lambda_e \otimes V_k$$

and refer to its inverse $\gamma_{\boldsymbol{k}}^{-1}$.

b) We have that

$$\gamma_{\boldsymbol{l}} W^{(m)}_{\boldsymbol{l}-\boldsymbol{i}/2,\boldsymbol{k}-\boldsymbol{i}/2}(x)\gamma_{\boldsymbol{k}}^{-1} = \Gamma_e^{(2m)}(x^{1/2}) \otimes V^{m^2}_{(l+1/4)^2,(k+1/4)^2}(x), \qquad (48)$$

where $V^h_{k_2,k_1}(x)$ is the Liouville vertex operator defined above using the map (35) with $s = 1/4$, which is an isomorphism for this value.

Proof. First, it is easy to see that the image of $\gamma_{\boldsymbol{k}}$ is the eigenspace of h_0 with eigenvalue $2k$, from which part a follows.

By propositions 2 and 3, we have

$$\gamma_{\boldsymbol{l}} W^{(m)}_{\boldsymbol{l}-\boldsymbol{i}/2,\boldsymbol{k}-\boldsymbol{i}/2}(x)\gamma_{\boldsymbol{k}}^{-1} = \Gamma_e^{(2m)}(x^{1/2}) \otimes x^{d_{\boldsymbol{k}}/2-d_{\boldsymbol{l}}/2}\Gamma_o^{(2m)}(x^{1/2}) \qquad (49)$$

Now we have

$$d_{\boldsymbol{l}}/2 - d_{\boldsymbol{k}}/2 = (l + 1/4)^2 - (k + 1/4)^2,$$

so we must show that

$$x^{-m^2}\Gamma_o^{(2m)}(x^{1/2})$$

satisfies (16). But this follows from proposition 1. To determine that the vaccuum expectation is one on both sides of (49), it is enough to check values on the lowest weight vectors, and notice that the partition in (34) is precisely the blended partition

$$(\mu, 0) = \mathcal{B}(\varnothing, \varnothing; k, -k).$$

$\square$

We now have

Corollary 1. *The AGT relations hold for $t_1 + t_2 = 0$.*

Proof. For an N-tuple of integers $\tilde{k}$, let

$$\boldsymbol{k}_i = (k_i, -k_i), \quad a_i = k_i + 1/4, \quad \boldsymbol{a}_i = (a_i, -a_i)$$

so that

$$W^{(m)}_{\boldsymbol{a}_i, \boldsymbol{a}_j}(x) = W^{(m)}_{\boldsymbol{k}_i - \boldsymbol{i}/2, \boldsymbol{k}_j - \boldsymbol{i}/2}(x),$$

by subtracting the constant $3/4$ from both the entries of $\boldsymbol{a}_i, \boldsymbol{a}_j$. It suffices to prove the claim for these values because the coefficients of q_i are rational functions, which are determined by their values at these points.

Now apply part b of the theorem to (43) to get

$$Z(\tilde{\boldsymbol{a}}, \tilde{m}, \tilde{q}) = \operatorname{Tr} q^d W^{(m_N)}_{\boldsymbol{k}_1 - \boldsymbol{i}/2, \boldsymbol{k}_N - \boldsymbol{i}/2}(x_N) \cdots W^{(m_1)}_{\boldsymbol{k}_2 - \boldsymbol{i}/2, \boldsymbol{k}_1 - \boldsymbol{i}/2}(x_1) =$$

$$\left(\operatorname{Tr}_{\Lambda_e} q^{d/2} \Gamma^{(2m_N)}_e(x_N^{1/2}) \cdots \Gamma^{(2m_1)}_e(x_1^{1/2}) \right) \times$$

$$\left(\operatorname{Tr} q^d V^{m_N^2}_{(k_1+1/4)^2, (k_N+1/4)^2}(x_N) \cdots V^{m_1^2}_{(k_2+1/4)^2, (k_1+1/4)^2}(x_1) \right).$$

The second factor is by definition $\mathcal{B}(\tilde{k}, \tilde{h}, \tilde{q})$ under the change of variables in (19), so it remains to show that the first factor equals $Z'(\tilde{m}, \tilde{q})$. This can be calculated using the commutation relations (26) and (27). We will verify it for $N = 1$, leaving the general case as an exercise:

$$\operatorname{Tr}_{\Lambda_e} q^{d/2} \Gamma^{(2m)}_e(x^{1/2}) = \operatorname{Tr} q^{d/2} \Gamma^{2m}_{-,e}(x^{1/2}) \Gamma^{-2m}_{+,e}(x^{-1/2}) =$$

$$\operatorname{Tr} \Gamma^{2m}_{-,e}((xq)^{1/2}) q^{d/2} \Gamma^{-2m}_{+,e}(x^{-1/2}) =$$

$$\operatorname{Tr} q^{d/2} \Gamma^{-2m}_{+,e}(x^{-1/2}) \Gamma^{2m}_{-,e}((xq)^{1/2}) =$$

$$(1 - q)^{2m^2} \operatorname{Tr} q^{d/2} \Gamma^{2m}_{-,e}((xq)^{1/2}) \Gamma^{-2m}_{+,e}(x^{-1/2}) =$$

$$\cdots$$

$$(q; q)^{2m^2}_{\infty} \operatorname{Tr} q^{d/2} \Gamma_{+,e}(x^{-1/2})$$

Since Γ_+ is unitriangular with respect to the degree grading, we get

$$(q; q)^{2m^2}_{\infty} \operatorname{Tr}_{\Lambda_e} q^{d/2} = (q; q)^{2m^2-1}_{\infty} = Z'(m, q).$$

$\square$

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